The Groupoid CwF of Containers

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Overview

CwFs in Intensional Type Theory

What are CwFs?

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A model constitutes a **sound semantics** for a type theory.

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Categories with families (CwFs) are one way to model dependent type theory.

If we write down the intrinsic syntax of dependent type theory as a quotient inductive-inductive type (QIIT), algebras of this signature correspond to CwFs.

Categories with families

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- A context extension operation $\Gamma.A$ for $\Gamma : |\mathbf{C}|$ and $A : \mathrm{Ty}\ \Gamma$ such that
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When working in intensional type theory (ITT) i.e. no UIP, in both cases, $\mathrm{Ty}\,\Gamma$ forms a **groupoid** not a **set**.

How do we solve this?

Force models to fit original CwF definition

↓

Strictification

Define a ‘higher’ CwF e.g. ∞-CwF, groupoid CwF

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Prove extra coherence laws

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Strictification

CwF of strictified containers

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e.g. ∞-CwF, groupoid CwF

Prove extra coherence laws

Groupoid CwF of containers

we do both

Related work on higher CwFs

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  We focus on groupoid CwFs.

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In a groupoid CwF (GCwF),

$$\mathrm{Ty} \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Gpd}$$

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Additional coherence laws on types need to checked.

$$\begin{tikzcd}
 & & A[\sigma][\delta][\gamma] \arrow[dll, Rightarrow, "{[\circ]\,\mathrm{Ty}}"'] \arrow[drr, Rightarrow, "{\mathrm{ap}_{-[\delta]}[\circ]\,\mathrm{Ty}}"] & & \\
A[\sigma][\delta \circ \gamma] \arrow[dr, Rightarrow, "{[\circ]\,\mathrm{Ty}}"'] & & & & A[\sigma \circ \delta][\gamma] \arrow[dl, Rightarrow, "{[\circ]\,\mathrm{Ty}}"] \\
 & A[\sigma \circ (\delta \circ \gamma)] \arrow[rr, Rightarrow, "{\mathrm{ap}_{A[-]}\,\mathrm{assoc}_{\mathbf{C}}^{-1}}"'] & & A[(\sigma \circ \delta) \circ \gamma] &
\end{tikzcd}$$

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A Groupoid CwF of Containers

Containers (a.k.a. polynomial functors)

Definition

A **(set)-container** is a pair $S : \mathrm{Set}, P : S \to \mathrm{Set}$ written $S \triangleleft P$.
Every container has a functor representation

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We are interested in a container model of type theory for reasons to do with semantics of inductive types.

The container model (as outlined in [Altenkirch and Kaposi, 2021])

- The category of contexts and substitutions is the category of set-containers $S_\Gamma : \mathrm{Set} \triangleleft P_\Gamma : S_\Gamma \to \mathrm{Set}$ and their morphisms. Set-containers have functors

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$$\int_{X:\mathrm{Set}} (\gamma : [\![\Gamma]\!]\, X) \to [\![A]\!]^G(X, \gamma)$$

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- Context extension is given by $\Gamma.A = S_A \triangleleft P_A^X$

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Terms	$\int_{X : \lvert\mathbf{C}\rvert} (\gamma : \Gamma\, X) \to A(X, \gamma)$	$\int_{X : \mathrm{Set}} (\gamma : [\![\Gamma]\!]\, X) \to [\![A]\!]^{G}(X, \gamma)$

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Context extension	$\Gamma.A\, X = \sum_{\rho:\Gamma\, X} (A\,(X, \rho))$	$[\![\Gamma.A]\!]\, X = \sum_{\rho:[\![\Gamma]\!]\, X} ([\![A]\!]^G\,(X, \rho))$

The container model — coherence issues

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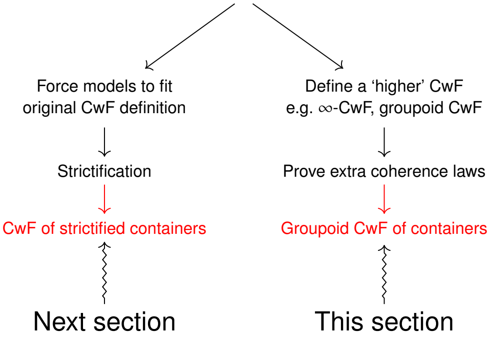
Recall . . .



The container model — contexts & types

Contexts If Γ is a context, then $\Gamma = S_\Gamma \triangleleft P_\Gamma$ is a **set-container**.

A substitution $\Delta \xrightarrow{\gamma} \Gamma$ is a container morphism:

$$\gamma_s : S_\Delta \to S_\Gamma$$
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Types If A : Ty Γ, then $A = S_A \triangleleft^G P_A$ is a **generalised container**, with S_A : Set, $P_A : S_A \to |\int [\![\Gamma]\!]|$, where we can break apart P_A into 3 components.

$$S_A : \mathrm{Set}$$
$$P_A^X : S_A \to \mathrm{Set}$$
$$P_A^s : S_A \to S_\Gamma$$
$$P_A^f : \prod_{s : S_A} P_\Gamma\,(P_A^s\, s) \to P_A^X\, s$$

The container model — type substitution

If $\Delta \xrightarrow{\gamma} \Gamma$ is a container morphism, then:

$$\begin{aligned} \gamma_s &: S_\Delta \to S_\Gamma \\ \gamma_p &: \prod_{s_\Delta : S_\Delta} P_\Gamma\,(\gamma_s\, s_\Delta) \to P_\Delta\, s_\Delta \end{aligned}$$

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Given a container morphism $\Delta \xrightarrow{\gamma} \Gamma$, we (roughly) define $A[\gamma]$ as:

$$\begin{tikzcd} S_{A[\gamma]} \arrow[r, "s_A"] \arrow[d, "s_\Delta"'] \arrow[dr, phantom, "\lrcorner", very near start] & S_A \arrow[d, "P_A^s"] & \triangleleft^G & P_\Gamma\,(\gamma_s\, s_\Delta) \arrow[r, "(P_A^f\, s_A)"] \arrow[d, "\gamma_p\, s_\Delta"'] & P_A^X\, s_A \arrow[d, "\mathrm{inr}"] \\ S_\Delta \arrow[r, "\gamma_s"'] & S_\Gamma & & P_\Delta\, s_\Delta \arrow[r, "\mathrm{inl}"'] \arrow[ur, phantom, "\ulcorner", very near end] & P_{A[\gamma]}^X\, s_{A[\gamma]} \end{tikzcd}$$

A trick for pullback in type theory

$$\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd}$$

can be written as

$$F : B \to \mathcal{U}$$
$$F\,b = \sum_{a:A} f\,a \equiv b.$$

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Then $$\begin{tikzcd} PB \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ C \arrow[r, "g"'] & B \end{tikzcd}$$ can be written as $$\begin{aligned} &H : C \to \mathcal{U} \\ &H\, c = F\,(g\, c) \\ &\quad = \sum_{a:A} f\, a \equiv g\, c. \end{aligned}$$

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So we represent pullbacks as families.

We don't have a trick for pushouts

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d] \\ Z \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & PO \end{tikzcd}$$

is written as $\|\text{Pushout}\, f\, g\|_0$, where

$$\begin{aligned} \mathsf{inl} &: Z \to \text{Pushout}\, f\, g \\ \mathsf{inr} &: Y \to \text{Pushout}\, f\, g \\ \mathsf{push} &: \prod_{x:X} \mathsf{inl}\,(f\,x) \equiv \mathsf{inr}\,(g\,x) \end{aligned}$$

and

$$\begin{aligned} |\_|_0 &: A \to \|A\|_0 \\ \mathsf{squash}_0 &: \prod_{x,y:\|A\|_0} \prod_{p,q:x\equiv y} p \equiv q. \end{aligned}$$

The container model — coherences

Triangulators (the identity coherence laws)

$$\begin{tikzcd} & A[\mathsf{id}_\Gamma][\gamma] \arrow[dl, Rightarrow, "{[\circ]\,\mathsf{Ty}}"'] \arrow[dr, Rightarrow, "{\mathsf{ap}_{-[\gamma]}[\mathsf{id}]\,\mathsf{Ty}}"] & \\ A[\mathsf{id}_\Gamma \circ \gamma] \arrow[rr, Rightarrow, "{\mathsf{ap}_{A[-]}\,\mathsf{idl}_{\mathbf{C}}}"'] & & A[\gamma] \end{tikzcd} \qquad \begin{tikzcd} & A[\gamma][\mathsf{id}_\Gamma] \arrow[dl, Rightarrow, "{[\circ]\,\mathsf{Ty}}"'] \arrow[dr, Rightarrow, "{[\mathsf{id}]\,\mathsf{Ty}}"] & \\ A[\gamma \circ \mathsf{id}_\Gamma] \arrow[rr, Rightarrow, "{\mathsf{ap}_{A[-]}\,\mathsf{idr}_{\mathbf{C}}}"'] & & A[\gamma] \end{tikzcd}$$

where

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I will talk about the left coherence law.

Left identity coherence law, in `Cubical Agda`

To prove:

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Proof sketch: Thanks to Axel's help!
We write the above as a square of generalised containers

$$\begin{tikzcd} A[\mathrm{id}_\Gamma \circ \gamma] \arrow[r, Rightarrow, "{\mathsf{ap}_{A[-]}\ \mathsf{idl}_{\mathbf{C}}}"] & A[\gamma] \\ A[\mathrm{id}_\Gamma][\gamma] \arrow[u, Rightarrow, "{[\circ]\ \mathsf{Ty}}"] \arrow[r, Rightarrow, "{\mathsf{ap}_{-[\gamma]}\,[\mathrm{id}]\ \mathsf{Ty}}"'] & A[\gamma] \arrow[u, Rightarrow, "\mathsf{refl}"'] \end{tikzcd}$$

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which we can rewrite as

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\begin{tikzcd}
A[\gamma] \arrow[r, Rightarrow, "\mathrm{uaGenCon}\ \ \mathrm{id}_{\simeq\mathrm{GenCon}}"] & A[\gamma] \\
A[\mathrm{id}_\Gamma][\gamma] \arrow[u, Rightarrow, "\mathrm{uaGenCon}\ (...[\circ]\ \mathrm{Ty}\text{-eq})"] \arrow[r, Rightarrow, "\mathrm{uaGenCon}\ (...[\mathrm{id}]\ \mathrm{Ty}\text{-eq})"'] & A[\gamma] \arrow[u, Rightarrow, "\mathrm{uaGenCon}\ \ \mathrm{id}_{\simeq\mathrm{GenCon}}"']
\end{tikzcd}
$$

where uaGenCon : $A \simeq_{\mathrm{GenCon}} B \to A \equiv B$.

Left identity coherence law, in `Cubical Agda`

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A[\gamma] \arrow[rr, Rightarrow, "\mathsf{uaGenCon}\ \mathrm{id}_{\simeq\mathrm{GenCon}}"] & & A[\gamma] \\
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We show that this square commutes by giving equalities for each of the generalised container components.

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where uaGenCon : $A \simeq_{\text{GenCon}} B \to A \equiv B$.

We show that this square commutes by giving equalities for each of the generalised container components.

Finally, to get the original square, we massage our equalities to match those we need via some lemmas.
E.g. one lemma is uaGenCon $\mathrm{id}_{\simeq\text{GenCon}} \equiv \text{refl}$. □

Progress so far

- Formalised left and right identity coherence laws in `Cubical Agda` i.e. the **triangulators**

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- Formalised left and right identity coherence laws in `Cubical Agda` i.e. the **triangulators**
- To do: associativity coherence law i.e. the **pentagonator**

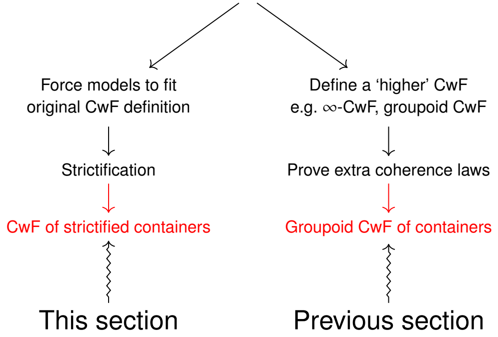
$$
\begin{tikzcd}
 & & A[\sigma][\delta][\gamma] \arrow[dll, Rightarrow, "{[\circ]\ \mathsf{Ty}}"'] \arrow[drr, Rightarrow, "{\mathsf{ap}_{-[\delta]}\,[\circ]\ \mathsf{Ty}}"] & & \\
A[\sigma][\delta \circ \gamma] \arrow[dr, Rightarrow, "{[\circ]\ \mathsf{Ty}}"'] & & & & A[\sigma \circ \delta][\gamma] \arrow[dl, Rightarrow, "{[\circ]\ \mathsf{Ty}}"] \\
 & A[\sigma \circ (\delta \circ \gamma)] \arrow[rr, Rightarrow, "{\mathsf{ap}_{A[-]}\ \mathsf{assoc}_{\mathbf{C}}^{-1}}"'] & & A[(\sigma \circ \delta) \circ \gamma] &
\end{tikzcd}
$$

A CwF of
Strictified Containers

Another way to solve the coherence issues

This section illustrates another way to deal with the coherence issues in the model given by [Altenkirch and Kaposi, 2021].

Recall . . .



The strictified container model

We use an inductive-recursive universe $\mathsf{U} : \mathsf{Set}, \mathsf{El} : \mathsf{U} \to \mathsf{Set}$.

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- The category of <u>contexts</u> and <u>substitutions</u> is the category of **codes** for set-containers and codes for their morphisms:

$$\hat{\Gamma} = \hat{S}_\Gamma : \mathsf{U} \triangleleft \hat{P}_\Gamma : \mathsf{El}\ \hat{S}_\Gamma \to \mathsf{U}$$

such that $\Gamma = S_\Gamma \triangleleft P_\Gamma = \mathsf{El}\ \hat{S}_\Gamma \triangleleft (\mathsf{El} \circ \hat{P}_\Gamma)$.

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  such that $\Gamma = S_\Gamma \lhd P_\Gamma = \mathrm{El}\ \hat{S}_\Gamma \lhd (\mathrm{El} \circ \hat{P}_\Gamma)$.

- <u>Types</u> in context $\hat{\Gamma} = \hat{S}_\Gamma \lhd \hat{P}_\Gamma$ are codes for generalised containers over $\int[\![\Delta]\!]$ for some context $\hat{\Delta}$, together with a substitution into $\hat{\Delta}$ — **we delay substitution**.

$$\begin{aligned} \hat{A} = (&\hat{\Delta} : |\mathbf{Con}|, \\ &\hat{\Gamma} \xrightarrow{\delta} \hat{\Delta}, \\ &\hat{S}_B : U \lhd \hat{P}_B : El\ \hat{S}_B \to |\textstyle\int[\![\Delta]\!]|^U) \end{aligned}$$

  Idea: $\hat{A}$ represents $\hat{B}[\delta]$.

The strictified container model

$$\hat{A} = (\hat{\Delta} : |\mathbf{Con}|, \\ \hat{\Gamma} \xrightarrow{\delta} \hat{\Delta}, \\ \hat{S}_B \lhd \hat{P}_B)$$

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$$\begin{aligned}\hat{A} = (\hat{\Delta} &: |\mathbf{Con}|,\\ \hat{\Gamma} &\xrightarrow{\delta} \hat{\Delta},\\ \hat{S}_B &\lhd \hat{P}_B)\end{aligned}$$

- Type substitution for $\gamma \colon \hat{\Theta} \to \hat{\Gamma}$, $\hat{A}[\gamma]$ can now be defined as

$$\hat{A}[\gamma] \coloneqq (\hat{\Theta}, \hat{\Theta} \xrightarrow{\delta \circ \gamma} \hat{\Delta}, \hat{S_B} \lhd \hat{P_B}).$$

The strictified container model

$$\hat{A} = (\hat{\Delta} : |\mathbf{Con}|,$$
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$$\hat{A}[\gamma] \coloneqq (\hat{\Theta}, \hat{\Theta} \xrightarrow{\delta\circ\gamma} \hat{\Delta}, \hat{S_B} \lhd \hat{P_B}).$$

The collection of types is a set, since every component of a type $\hat{A}$ is of type U instead of Set. This fits the original CwF definition.

In conclusion

- In a setting without UIP (like in HoTT), CwFs raise coherence issues
- 2 ways to solve this: by defining 'higher' CwFs, or by strictifying
- We focus on a specific example: the container model
- 1^{st} approach is ongoing work: proving higher coherence laws for the container GCwF, formalised in `Cubical Agda`
- 2^{nd} approach involves using an inductive-recursive universe and delaying type substitution

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Thank you!

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